Varianta 072

Subiectul I

a) $\sqrt{41}$. b) d(E,d)= $\sqrt{5}$. c) y = x + 2 d) $A=\frac{3}{2}$; e) $\cos(\vec{v},\vec{w})=\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}=\frac{-\sqrt{5}}{5}$;

e) $\{a=1,\ b=-6$

Subiectul II

1. a) In Z_5 $\hat{2}^{4k}=\hat{1},\forall k\in\mathbf{N}\Rightarrow\hat{2}^{2007}=\hat{2}^3=\hat{3}$. b) Probabilitatea este $\frac{2}{7}$. c) $n=5$. d) $x=1$.

e) $\sum_{i=1}^{n}\left(x_i^2\right)=\left(\sum_{i=1}^{n}x_i\right)^2-2\left(\sum_{1\le i<i\le n}x_iy_i\right)=2$.

2. a) $f'(x)=2\cos x-3\sin x$. b) $\int_0^{\pi}f(x)dx=4$.

c) $f''(x)\le 0,(\forall)\in\left[0,\frac{\pi}{2}\right]\Rightarrow f$ este concava

d) $\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=2\cos1-3\sin1$; e) $\lim_{x\to\infty}\frac{f(x)}{x}=0$.

Subiectul III

a) Se verifică prin calcul simplu relațiile: $\overline{z+w}=\bar{z}+\bar{w}$ şi $\overline{z\cdot w}=\bar{z}\cdot\bar{w},\forall z,w\in\mathbf{C}$

$f(X+Y)=(\overline{X+Y})=\overline{X}+\overline{Y}=f(X)+f(Y)$

$f(X\cdot Y)=\overline{X\cdot Y}=\overline{X}\cdot\overline{Y}=f(X)\cdot f(Y),\forall X,Y\in\mathrm{M}_2(\mathbf{C})$

c) $(f\circ f)(X)=f(f(X))=\overline{f(\overline{X})}=(\overline{\overline{X}})=X,\forall X\in\mathrm{M}_2(\mathbf{C})$

d) din c) rezulta $f^{-1}=f$

e)

$g_A(X+Y)=A(X+Y)A^{-1}=(AX+AY)A^{-1}=AXA^{-1}+AYA^{-1}=g_A(X)+g_A(Y),\forall X,Y\in\mathrm{M}_2(\mathbf{C})$.

$g_A(XY)=A(XY)A^{-1}==AX\left(A^{-1}A\right)YA^{-1}=\left(AXA^{-1}\right)\left(AYA^{-1}\right)=g_A(X)g_A(Y),\forall X,Y\in\mathrm{M}_2(\mathbf{C})$.

f) Deoarece $\forall X,Y\in\mathrm{M}_2(\mathbf{C})$ astfel incat $g_A(X)=g_A(Y)\Leftrightarrow AXA^{-1}=AYA^{-1}$ obtinem X=Y (inmultim egalitatea cu A^{-1} la stanga si cu A la dreapta) si $\Rightarrow$ ca g_A este injectiva si pentru

$\forall Y\in\mathrm{M}_2(\mathbf{C})$ astfel incat $g_A(X)=Y\Rightarrow X=A^{-1}YA\in\mathrm{M}_2(\mathbf{C})\Rightarrow g_A$ este surjectiva, deci bijectiva.

g) Reducere la absurd: Presupunem ca $\exists A\in\mathrm{M}_2(\mathbf{C})$ astfel incat f(X) = $g_A(X)$,

$\forall X\in\mathrm{M}_2(\mathbf{C})\Leftrightarrow\overline{X}=AXA^{-1}$. Alegem $X=iI_2\in\mathrm{M}_2(\mathbf{C})\Rightarrow\overline{iI_2}=iAI_2A^{-1}\Leftrightarrow$

$-iI_2=iI_2$, fals.

Subiectul IV

a) Pentru n=0 din relația de recurență obținem $f_1(x)=\int_0^x f_0(t)dt\Rightarrow f_1(x)=e^x-1$.

b) Pentru n=1 $\Rightarrow f_2(x)=\int_0^x f_1(t)dt=\int_0^x\left(e^t-1\right)dt=(e^t-t)\Big|_0^x=e^x-x-1$.

c) $\lim\limits_{x\to-\infty} f_0(x) = \lim\limits_{x\to-\infty} e^x = 0 \in \mathbf{R} \Rightarrow y = 0$ asimptota orizontala la G_f la $-\infty$.

d) Notam cu $P(n): f_{n+1}(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - ... - \frac{x^n}{n!}, \forall n \in \mathbf{N}, \forall x \in \mathbf{R}$.

Deoarece $P(0): f_1(x) = e^x - 1$ adevarata (din a)) mai ramane de aratat ca din P(k)(a) $\Rightarrow$

P(k+1) (a), unde P(k): $f_{k+1}(x) = e^x - 1 - \sum_{i=1}^{k} \frac{x^i}{i!}$ este adevarata si

P(k+1): $f_{k+2}(x) = e^x - 1 - \sum_{i=1}^{k+1} \frac{x^i}{i!}$. Din relatia de recurenta

$$\Rightarrow f_{k+2}(x) = \int_0^x f_{k+1}(t)dt = \int_0^x \left(e^x - 1 - \sum_{i=1}^{k} \frac{x^i}{i!} \right) dx = \left(e^t - t - \frac{t^2}{2!} - ... - \frac{t^{k+1}}{(k+1)!} \right) \Bigg|_0^x \Rightarrow P(k+1) \text{ adevarat}$$

$\Rightarrow \quad P(n)$ adevarat, $(\forall) n \in \mathbf{N}$.

e)

$I.$ $P(n): 0 < f_n(x), \forall n \in \mathbf{N}, \forall x > 0.$ Verificam ca $P_0: 0 < f_0(x)$ (a) si din P(k) (a) $\Rightarrow$ P(k + 1)

unde P(k) : $0 < f_n(x)(a)\ \forall x \in (0, \infty)$. Deoarece $f_{k+1}(x) = \int_0^x f_k(t)dt > 0$ (din P(k) (a)) $\Rightarrow$

P(n) (a) $\forall x > 0$ si $\forall n \in \mathbf{N}$.

$II.$ Notam P(n) : $f_n(x) \le e^x \frac{x^n}{n!}, \forall n \in \mathbf{N}, \forall x > 0$. P(0) : $f_0(x) \le e^x \Leftrightarrow e^x \le e^x$ (a), si din

P(k) : $f_k(x) \le e^x \frac{x^k}{k!}$ avem de aratat P(k + 1) : $f_{k+1}(x) \le e^x \frac{x^{k+!}}{(k+1)!} \Leftrightarrow$

$$\Leftrightarrow e^x - 1 - \frac{x}{1!} - ... - \frac{x^{k-1}}{(k-1)!} - \frac{x^k}{k!} \le e^x \frac{x^{k+1}}{(k+1)!} \Leftrightarrow f_k(x) - \frac{x^k}{k!} \le e^x \frac{x^{k+1}}{(k+1)!}.$$

Fie $h:(0,\infty)\to\mathbf{R}, h(x)=f_{k+1}(x)-e^{x}\frac{x^{k+1}}{(k+1)!}\Rightarrow l_d(0)=\lim_{x\to0}h(x)=0$ si

$h'(x)=f_k(x)-e^{x}\frac{x^{k}}{k!}-e^{x}\frac{x^{k+1}}{(k+1)!}<0,\forall x>0\Rightarrow$ h str. cresc. pe $(0,\infty)\Rightarrow h(x)\le 0,\forall x>0\Rightarrow$

$\Rightarrow f_{k+1}(x)\le e^{x}\frac{x^{k+1}}{(k+1)!}\Rightarrow P(n)$ este adevarata pentru orice $n\in\mathbf{N},\forall x>0$.

f) Din e) $\Rightarrow 0<e^{x}-1-\frac{x}{1!}-...-\frac{x^{n}}{n!}\le e^{x}\frac{x^{n+1}}{(n+1)!}$. Deoarece $\lim_{n\to\infty}e^{x}\frac{x^{n+1}}{(n+1)!}=0\Rightarrow$

$\Rightarrow\lim_{n\to\infty}\left[e^{x}-\left(1+\frac{x}{1!}+...+\frac{x^{n}}{n!}\right)\right]=0\Rightarrow\lim_{n\to\infty}\left(1+\frac{x}{1!}+...+\frac{x^{n}}{n!}\right)=e^{x}$.

$g)\ f_{2n+1}(x)=e^{x}-1-\sum_{k=12}^{2n}\frac{x^{k}}{k!}\Rightarrow f_{2n+1}(1)+f_{2n+1}(-1)=e+\frac{1}{e}-2\left(1+\frac{1}{2!}+...+\frac{1}{(2n)!}\right)\Rightarrow$

$\Rightarrow$ limita cautata este $L=\frac{e+e^{-1}}{2}$.